Experimental Direct Observation of Mixed State Entanglement

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We report on the direct estimation of concurrence for mixed quantum states. The used method relies on joint measurements on two copies of an entangled state. In the experimental demonstration two polarization-entangled photon pairs emitted from spontaneous parametric down-conversion are analyzed together using a linear optics controlled phase gate. We demonstrate that the measured data, without need for further numerical processing, directly yield reliable estimates, despite experimental imperfections.

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The experimental techniques for the preparation of entangled states have been improved substantially during the last years, such that entangled states can be prepared in an increasing variety of physical systems. Also the number of subsystems that can be entangled is growing rapidly [1–3]. Therefore, efficient experimental strategies to verify the preparation of entanglement, quantitatively *and* qualitatively become more and more important.

Quantum state tomography (QST) has proven useful for comparatively small systems [4]. However, as the involved measurement effort scales exponentially with increasing system size, it has already reached its limitations [3], and is thus not a viable choice on the way to large scale entanglement. Other techniques, like Bell inequalities [5,6] and entanglement witnesses [7] are much more suitable for the analysis of multipartite entanglement [2] since the number of observables to be measured increases significantly slower [8–10]. Still, these tools have the disadvantage that a particular choice of them applies only to a rather small set of entangled states.

An alternative path are measurements on multiple identically prepared quantum systems [11–13]. This allows the direct observation of separability criteria and entanglement measures but has proven to be significantly more efficient than measurements in the traditional setting of single quantum systems [14–16]. For example, as was shown recently [17–19], the concurrence $\mathcal{C}$ [20–22] of *pure* quantum states can be derived from a simple parity measurement on two copies of a quantum states.

The intricacy of entangled states, however, comes with mixing. For *mixed* states, which are rather common in experiments, the concurrence cannot be defined in terms of a simple measurement prescription. It is defined via an optimization over all pure state decompositions of the mixed state under consideration [23]. Such optimizations are complex mathematical problems and there is no prospect to find exact general solutions beyond the case of two qubits [21]. In particular, it seems illusive to redefine concurrence of mixed states in terms of a simple measurement prescription. However, *lower bounds* on the concurrence of arbitrary mixed states [24,25] can be measured in the presently discussed framework of two identically prepared quantum states.

Here we report on the direct experimental observation of this bound for photonic qubits. Thereby, we make neither assumptions on the state's purity, nor on the faithfulness of preparation of identical quantum states. Before we describe the experimental implementation of the measurement prescription let us shortly recapitulate the underlying theory derived in [17,24,25].

Concurrence of a pure state $|\Psi\rangle$ can be defined as

$$\mathcal{C}(\Psi) = 2\sqrt{\langle\Psi| \otimes \langle\Psi|\hat{A}|\Psi\rangle \otimes |\Psi\rangle}, \tag{1}$$

where $\hat{A}$ is the projector onto the antisymmetric components of the duplicate local Hilbert spaces [17,18], i.e. $\hat{A} = \hat{P}_-^{(A)} \otimes \hat{P}_-^{(B)}$, where (A) and (B) label the two subsystems. For qubits $\hat{P}_-$ takes the particularly simple form $\hat{P}_- = |\psi^-\rangle\langle\psi^-|$ with $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$; i.e., concurrence is given in terms of the probability $\wp_{--}$ to find both duplicate subsystems in the singlet state.

Although Eq. (1) might overestimate the concurrence for mixed states, it can be used to find a valid lower bound if the mixedness of the state is taken into account. The underlying idea is based on the fact that two identically prepared *pure* states form a globally symmetric object: if the first-subsystem components are observed in a symmetric state, then also the second-subsystem components will be found in a symmetric state, and, similarly, for the observation of an antisymmetric state. Consequently, the probability to find the components of the two different subsystems in states with different symmetry provides information on the mixing of the underlying state. Following these considerations, it has been shown that concurrence of an arbitrary mixed state ρ is bounded from below by $\mathcal{C}^2 \geq 2(\wp_{--} - \wp_{+-})$ as well as by $\mathcal{C}^2 \geq 2(\wp_{--} - \wp_{-+})$ [24]. Thereby $\wp_{+-}$ is the probability to observe the first-subsystem of $\rho \otimes \rho$ in a symmetric state, and the second-subsystem in an antisymmetric state

$$\wp_{+-} = \mathrm{tr}((\hat{P}_+^{(A)} \otimes \hat{P}_-^{(B)})\rho \otimes \rho), \tag{2}$$

and analogously for $\wp_{-+}$ with $\hat{P}_+ = \mathbb{1} - \hat{P}_-$.

So far, it was assumed that the required two quantum states can indeed be prepared in exactly the same fashion, what poses a challenge to an experimental implementation. However, the approach described above directly translates also to an experiment in which two different quantum states ρ^1 and ρ^2 are prepared. In that case, the product of the concurrences of the two states is bounded from below by [25],

$$\mathcal{C}(\rho^1)\mathcal{C}(\rho^2) \geq B_i = \mathrm{tr}(\rho^1 \otimes \rho^2 \hat{V}_i), \tag{3}$$

with $\hat{V}_1 = 4(\hat{P}_-^{(A)} - \hat{P}_+^{(A)}) \otimes \hat{P}_-^{(B)}$ and $\hat{V}_2 = 4\hat{P}_-^{(A)} \otimes (\hat{P}_-^{(B)} - \hat{P}_+^{(B)})$.

In the following, we will discuss the experimental implementation of this measurement and give evidence that the lower bound provides an accurate estimate of the actual value of concurrence for states that are mixed due to imperfections of state of the art experiments.

In our experimental implementation the qubits of the considered states are encoded in the polarization of single photons propagating in well-defined spatial modes. The projection measurement of the components of the subsystems onto the symmetric or antisymmetric state space is accomplished by the use of a linear optics logic gate. The gate relies on the second order interference of two photons at a partially polarizing beam splitter. Its detailed description and characterization can be found in [26].

The entangled photon states are prepared by type II spontaneous parametric down-conversion (SPDC). A 2 mm thick β-Barium Borate (BBO) crystal is pumped by UV pulses with a central wavelength of 390 nm and an average power of 700 mW from a frequency-doubled mode-locked Ti:sapphire laser (pulse length 130 fs). The source is aligned such that the state $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle)$ is emitted. In order to obtain initially two copies of this state, the SPDC is operated in a double pass configuration leading to two pairs emitted in four spatial modes a, b and c, d (see Fig. 1) [1]. The modes are well defined by coupling the photon pairs into single mode fibers. The two down-conversion sources are aligned in identical ways to ensure that they provide indeed two copies of the same state within experimental uncertainties. The UV-mirror reflecting the pump beam in the backward direction is positioned 3 cm behind the down-conversion crystal. The small distance (compared to 40 cm Rayleigh length of the pump beam) together with a proper tilt of the mirror leads to identical phase-matching conditions for both SPDC

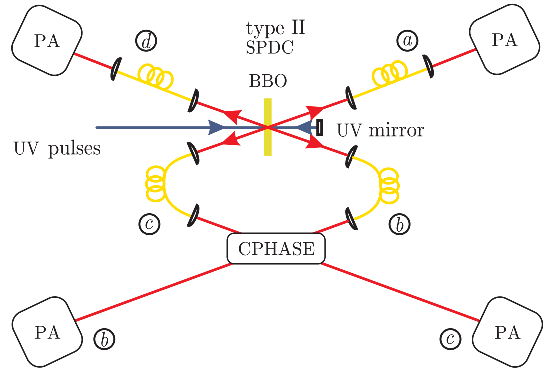

FIG. 1 (color online). Experimental setup for the direct measurement of concurrence. The two copies of the state are provided by two Bell-pairs originating from type II spontaneous parametric down-conversion (SPDC) processes in the spatial modes a, b and c, d. After passing the crystal, the beam is reflected back by a UV mirror. If a photon pair is created in each of the two passages, two copies of an entangled state are obtained. Half- and quarter wave plates in conjunction with polarizing beam splitters are used for the polarization analysis (PA). The projection of the subsystem parts onto the symmetric and antisymmetric state space is achieved by a measurement in the Bell basis using a controlled phase gate (CPHASE).

emissions. The single mode fiber couplers are aligned to select the same spectral range of down-converted photons for all four modes. The spectra are further equalized by narrow bandwidth interference filters ($\Delta\lambda = 3$ nm in modes a, b and $\Delta\lambda = 2$ nm in modes b, c). We perform polarization analysis in each of the spatial modes by a polarizing beam splitter and half- and quarter wave plates. The signals of the single photon counters (silicon avalanche photo diodes) are fed into a multichannel coincidence unit which allows to register any possible coincidence between the eight detectors. After complete alignment procedure we achieve polarization correlation visibilities of ~98% (98%) in the computational basis and ~94% (95%) in the conjugate diagonal basis for the entangled photon pairs emitted in the forward (backward) direction.

The projection measurement onto the distinct symmetry components is accomplished by a complete Bell state measurement of modes b and c. The symmetric subspace is thereby spanned by the Bell states $\{|\phi^\pm\rangle, |\psi^+\rangle\}$, with $|\phi^-\rangle = \frac{1}{\sqrt{2}}(|HH\rangle - |VV\rangle)$, $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle)$, and $|\psi^-\rangle$ is the only antisymmetric state.

The circuit for the measurement procedure is drawn in Fig. 2. As can be seen, the CPHASE gate together with the Hadamard gates $\mathcal{H}_1$, $\mathcal{H}_3$ and $\mathcal{H}_4$ implements a Bell measurement using a CNOT. For practical reasons it is desirable not to have an operation acting on the qubit in mode c prior to the CPHASE gate. Therefore we substitute the required gate $\mathcal{H}_3$ in mode c by the gate $\mathcal{H}_5$ in mode d: Since $\mathcal{H}_2 \otimes \mathcal{H}_5$ leaves the initial state invariant, $|\phi^+\rangle = (\mathcal{H}_2 \otimes \mathcal{H}_5)|\phi^+\rangle$, the sequence of gates $\mathcal{H}_2$ and $\mathcal{H}_3$

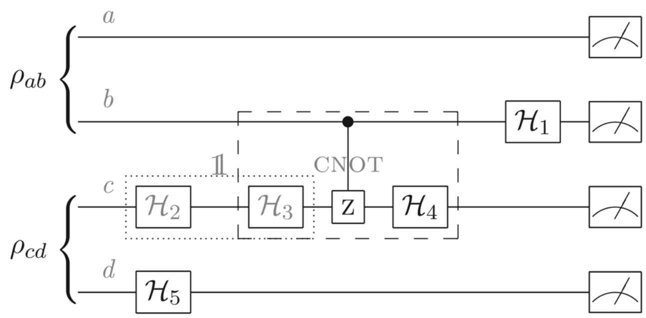

FIG. 2. Circuit diagram for the direct measurement of concurrence. In this configuration, the circuit yields an estimation for $\mathcal{C}(\rho_{ab})\mathcal{C}(\rho'_{cd})$ with $\rho' = (\mathcal{H}_2 \otimes \mathcal{H}_5)\rho_{cd}(\mathcal{H}_2 \otimes \mathcal{H}_5)^{\dagger}$.

equals the identity and does not need to be physically realized in the setup. Note that, for mixed states, a measurement of the probabilities $\wp_{--}$, $\wp_{+-}$, $\wp_{-+}$ in this configuration yields the lower bounds B_1 and B_2 on the product of the concurrences of both input states according to

$$\mathcal{C}(\rho_{ab})\mathcal{C}(\rho'_{cd}) \geq B_i \tag{4}$$

with $\rho'_{cd} = (\mathcal{H}_2 \otimes \mathcal{H}_5)\rho_{cd}(\mathcal{H}_2 \otimes \mathcal{H}_5)^{\dagger}$. The determination of these bounds in our setup requires only three correlation measurements, $\sigma_z \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x$, $\sigma_y \otimes \sigma_x \otimes \sigma_x \otimes \sigma_y$ and $\sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_z$, (with the Hadamard gates taken into account), where σ_x, σ_y, σ_z are the Pauli matrices.

The detection of photons in modes b and c in the state $|--\rangle$ corresponds to a projection onto the antisymmetric subspace [with $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$]. Thus, the share of these events in all detected coincidences gives the probability $\wp_{-} = \mathrm{tr}((\mathbb{1} \otimes \hat{P}_{-} \otimes \mathbb{1})(\rho_{ab} \otimes \rho'_{cd}))$ to find the subsystem (b, c) in the antisymmetric subspace. The remaining events where the photons in modes b and c are measured in the states $|++\rangle$, $|+-\rangle$ and $|-+\rangle$ lead to the probability to observe this subsystem in the symmetric subspace. Correlation measurements of the form $\sigma_z^a \otimes \square \otimes \square \otimes \sigma_x^d$, $\sigma_y^a \otimes \square \otimes \square \otimes \sigma_y^d$ and $\sigma_x^a \otimes \square \otimes \square \otimes \sigma_z^d$ (where $\square$ can be any Pauli matrix) allow the reconstruction of the fidelity on any of the four Bell states of the qubits a and d. The fidelity $\mathcal{F}_{\psi^-}$ to the state $|\psi^-\rangle$ equals thereby the probability to find the subsystem (a, d) in the antisymmetric subspace, whereas $1 - \mathcal{F}_{\psi^-}$ gives the probability to observe it in the symmetric subspace.

Finally, the evaluation of the quantities $\mathcal{F}_{\psi^-}$ and $1 - \mathcal{F}_{\psi^-}$ for the modes a and d with respect to the detection events in modes b and c yields all the *conditional* probabilities needed for the calculation of the concurrence according to Eqn. (3). For a measurement time of approximately 420 min for each setting this results in

$$\wp_{--} = 0.208 \pm 0.007, \qquad \wp_{-+} = 0.050 \pm 0.006, \qquad \wp_{+-} = 0.061 \pm 0.012, \tag{5}$$

leading to the bounds

$$\mathcal{C}(\rho_{ab})\mathcal{C}(\rho'_{cd}) \geq \begin{cases} \wp_{--} - \wp_{-+} & = 0.632 \pm 0.037 \\ \wp_{--} - \wp_{+-} & = 0.588 \pm 0.055 \end{cases}. \tag{6}$$

Given faithful preparation of two identical quantum states, i.e., $\rho_{ab} = \rho'_{cd} = \rho$ this yields the bound $\mathcal{C}(\rho) \geq 0.795$. Because of experimental imperfections this value is smaller than the maximum concurrence of 1, but the state can still reliably be detected as entangled. In situations in which one cannot rely on the exact preparation of identical quantum states, and in which no knowledge on the preparation of the second quantum state is available, one can obtain an estimate by a worst case assumption. Since, according to Eq. (4), $\mathcal{C}(\rho_{ab})$ is inversely proportional to $\mathcal{C}(\rho'_{cd})$, this amounts to considering $\mathcal{C}(\rho'_{cd})$ to be maximally entangled, what leads to the estimate

$$\mathcal{C}(\rho_{ab}) \geq \wp_{--} - \wp_{-+} = 0.632 \pm 0.037. \tag{7}$$

This unambiguously characterizes ρ_{ab} to be entangled. That is, even in the case of completely unreliable preparation of pairs of quantum states, the present measurement yields a accurate quantification of entanglement.

An important issue is to consider the influence of imperfect gate operation on the above discussed measurement procedure. For our gate, deviations from ideal operation are predominantly caused by imperfect interference which leads to vertically polarized noise in the output. The quantum channel corresponding to the experimental gate was characterized by a quantum process matrix [26]. Applying this quantum channel to a maximally entangled state yields the probabilities $\wp_{--} = 0.212$, $\wp_{-+} = \wp_{+-} = 0.038$ what gives rise to the measured bound $\mathcal{C}(\rho) \geq 0.834$ as com-

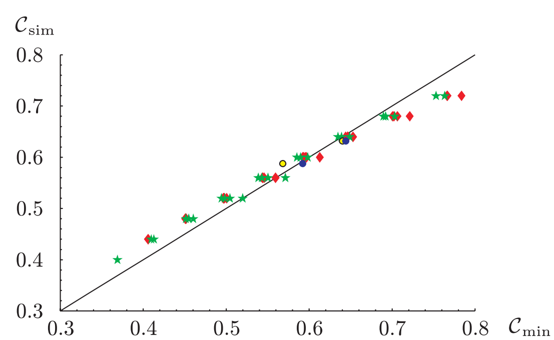

FIG. 3 (color online). Bounds on concurrence $\mathcal{C}_{\rm sim}$ obtained with the imperfect experimental phase gate versus the smallest bound on concurrence $\mathcal{C}_{\rm min}$ that would have been observed with a perfect gate, where the minimization is taken over all initial states that give rise to following measurement outcomes: ★, $\wp_{--}$, $\wp_{+-}$; ◆, $\wp_{--}$, $\wp_{-+}$; large solid circle, $\wp_{--} = 0.208$, $\wp_{-+} = 0.05$; medium solid circle, $\wp_{--} = 0.208$, $\wp_{+-} = 0.061$; solid circle, $\wp_{--} = 0.0208$, $\wp_{+-} = 0.061$, $\wp_{-+} = 0.05$.

pared to the actual value of 1. Obviously, the experimental imperfection of the gate has a significant influence on the estimation of entanglement and causes an underestimation of concurrence.

In order to get a quantitative estimate of this influence, we searched numerically for the least entangled pair of input states $\rho^1_{ab} \otimes \rho^2_{cd}$ for a given measurable bound of concurrence using the imperfect experimental gate. Figure 3 shows the simulated bound that is obtained with the experimental gate, $\mathcal{C}_{\text{sim}}$, as function of the smallest bound on concurrence with a perfect gate, $\mathcal{C}_{\text{min}}$, where the minimum has been taken over all pairs of initial states $\rho \otimes \rho'$, that yield measurement outcomes that are compatible with observed ones. Displayed are different constraints on the input states, e.g., to reproduce $\wp_{--}$ and $\wp_{+-}$ (green stars) or $\wp_{--}$ and $\wp_{-+}$ (red lozenges), respectively. The diagonal line divides the two regimes in which the imperfection of the gate leads to an overestimation (above the diagonal) or an underestimation of concurrence. For values of concurrence larger than $\sim\sqrt{0.65}$, that is, strongly entangled states whose preparation is desired, the imperfections of the controlled phase gate lead to a systematic underestimation of concurrence and thus to a reliable bound. Only for rather weakly entangled states the experimental imperfections might lead to a minor overestimation of concurrence. The least entangled state that still yields our experimentally measured values for $\wp_{--}$ and $\wp_{-+}$ has $\mathcal{C}_{\text{sim}} = 0.640$ which is slightly higher than $\mathcal{C}_{\text{min}} = 0.632$. In contrast, the value for $\wp_{--}$ and $\wp_{+-}$ of $\mathcal{C}_{\text{sim}} = 0.568$ is a little lower than $\mathcal{C}_{\text{min}} = 0.588$ (yellow bullets). If the imposed constraints are tightened such that the input state should reproduce not only the measured values $\{\wp_{--}, \wp_{+-}\}$ *or* $\{\wp_{--}, \wp_{-+}\}$, but $\wp_{--}, \wp_{+-}$ *and* $\wp_{-+}$, the values $\mathcal{C}_{\text{sim}} = 0.632$, $\mathcal{C}_{\text{min}} = 0.644$ for $\{\wp_{--}, \wp_{-+}\}$ and $\mathcal{C}_{\text{sim}} = 0.588$, $\mathcal{C}_{\text{min}} = 0.592$ for $\{\wp_{--}, \wp_{+-}\}$ (blue bullets) are obtained. Whereas one cannot strictly separate the influences from gate imperfections and imperfect state preparation, Fig. 3 shows that in the interesting regime of highly entangled states, an imperfect gate does *not* lead to an overestimation of the actual value of the concurrence, but yields a reliable lower bound.

In conclusion, we have shown that the present method provides a reliable estimate on concurrence, even given experimental imperfections. In particular, it does not depend on assumptions on a state's purity or faithfulness of preparation. Here, we assessed concurrence with only three measurement settings instead of nine for a full QST—such a decrease renders this approach particularly advantageous for deterministic state sources. Furthermore, QST often yields unphysical density matrices which are subjected to fitting routines in order to calculate the concurrence of the state. In contrast, here we rely directly on measurement results, rendering any intermediate steps unnecessary. Together with the moderate scaling for increasing system size [19], this approach paves the way towards the experimental investigation of ever more complex entangled states, even in systems where a complete set of observables is not accessible with current experimental technology.

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