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Four-photon entanglement from down-conversion

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Double-pair emission from type-II parametric down-conversion results in a highly entangled four-photon state. Due to interference, which is similar to bunching from thermal emission, this state is not simply a product of two pairs. The observation of this state can be achieved by splitting the two emission modes at beam splitters and subsequent detection of a photon in each output. Here we describe the features of this state and construct a Bell-type inequality, which gives a necessary and sufficient condition for a four-photon test of local realistic hidden variable theories.

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Parametric down-conversion has proven to be the best source of entangled photon pairs so far in an ever increasing number of experiments on the foundations of quantum mechanics [1] and in the new field of quantum communication. Experimental realizations of concepts like entanglement based quantum cryptography [2], quantum teleportation [3], and its variations [4] demonstrated the usability of this source. New proposals for quantum communication schemes [5] and, of course, for improved tests of local hidden variable theories initiated the quest for entangled multiphoton states. Interference of photons generated by independent down-conversion processes enabled the first demonstration of a three-photon Greenberger-Horne-Zeilinger (GHZ) argument [6] and, quite recently, even the observation of a four-photon GHZ state [7].

In this Rapid Communication we show that four-photon entanglement can be obtained directly from type-II parametric down-conversion. Instead of sophisticated but fragile interferometric setups, we utilize bosonic interference in a double-pair emission process. This effect causes strong correlations between measurement results of the four photons and renders type-II down-conversion a valuable tool for new multiparty quantum communication schemes. The analysis of the entanglement inherent in the four-photon emission leads us to a new form of inequality distinguishing local hidden variable theories from quantum mechanics, and demonstrates its potentiality for experiments on the foundations of quantum mechanics.

In type-II parametric down-conversion [8] multiple emission events during a single pump pulse lead to the following state:

$$Z\exp[-i\alpha(a_V^*b_H^*+a_H^*b_V^*)]|0\rangle, \tag{1}$$

where Z is a normalization constant, α is proportional to the pulse amplitude, and where a_V^* is the creation operator of a photon with vertical polarization in mode a, etc. (Fig. 1). We are interested only in four-photon effects, i.e., the emission of two pairs [9]. Then only the term in Eq. (1) proportional to

$$(a_H^*b_V^*+a_V^*b_H^*)^2|0\rangle \tag{2}$$

is relevant. The particle interpretation of this term can be obtained by its expansion

$$(a_H^{*2}b_V^{*2}+a_V^{*2}b_H^{*2}+2a_V^*a_H^*b_V^*b_H^*)|0\rangle, \tag{3}$$

and is given by the following superposition of photon number states:

$$|2H_a,2V_b\rangle+|2V_a,2H_b\rangle+|1H_a,1V_a,1H_b,1V_b\rangle, \tag{4}$$

where e.g., $2H_a$ means two H polarized photons in the beam a.

One should stress here that this type of description is valid only for down-conversion emissions, which are detected behind filters endowed with a frequency band, which is narrower than that of the pumping fields [10]. If a wide band down-conversion is accepted, then such a state is effective only if counts at the detectors are treated as coincidences, when they occur within time windows narrower than the inverse of the bandwidth of the radiation [11]. If such conditions are not met, then the four-photon events are essentially emissions of two independent, entangled pairs, with the entanglement existing only within each pair.

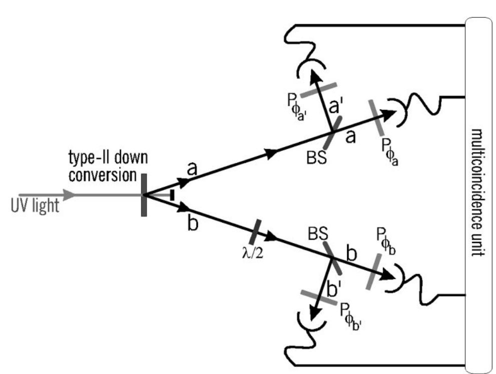

FIG. 1. Experimental setup to demonstrate the entanglement inherent in four-photon emission from type-II parametric down-conversion. ($\lambda/2$, half wave plate to flip polarization; BS, beam splitter; and P_{ϕ_a} represents polarization analysis corresponding to the phase angle ϕ_a, etc.)

Let us pass the four-photon state via two polarization independent 50-50 beam splitters. For simplicity we assume that at the beam splitter a is transformed into $(1/\sqrt{2})(a+a')$ and b into $(1/\sqrt{2})(b+b')$, with the prime denoting the reflected beam. One can expand the expression (4), and then extract only those terms that lead to four-photon coincidence behind the two beam splitters, i.e., only those terms for which there is one photon in each of the beams. The resulting component of the full state is given by

$$[4(a_H^* a'^*_H b_V^* b'^*_V + a_V^* a'^*_V b_H^* b'^*_H) + 2(a_H^* a'^*_V + a_V^* a'^*_H) \times (b_V^* b'^*_H + b_H^* b'^*_V)]|0\rangle. \tag{5}$$

The first term represents a four-photon GHZ state, whereas the second one is a product state of two Einstein-Podolsky-Rosen–Bohm states (the $|\Psi^+\rangle$ Bell states in polarizations H and V). This, after the normalization, can be symbolically written as

$$\sqrt{2/3}|\mathrm{GHZ}\rangle_{aa'bb'} + \sqrt{1/3}|\mathrm{EPR}\rangle_{aa'}|\mathrm{EPR}\rangle_{bb'}. \tag{6}$$

For additional simplicity of presentation we also rotate the polarizations in the beams a and a' by 90°. Thus now our initial state is given by Eq. (6) with the GHZ state in its standard form, resulting in the state

$$\sqrt{1/3}(|VVVV\rangle_{aa'bb'} + |HHHH\rangle_{aa'bb'} + \tfrac{1}{2}(|HVHV\rangle_{aa'bb'} + |HVVH\rangle_{aa'bb'} + |VHHV\rangle_{aa'bb'} + |VHVH\rangle_{aa'bb'})). \tag{7}$$

In order to demonstrate the entanglement of this state let us analyze polarization correlation measurements involving all four exit ports of the beam splitters, where the actual observables to be measured are elliptic polarizations with the main axis of the polarization ellipse at 45°. Such observables are of dichotomic nature, i.e., endowed with two valued spectra $k=+1,-1$, and are defined for each spatial propagation mode $x=a,a',b,b'$ by their eigenstates

$$\sqrt{1/2}|V\rangle_x + k e^{-i\phi_x}\sqrt{1/2}|H\rangle_x = |k,\phi_x\rangle. \tag{8}$$

The probability amplitudes for the results $k,l,m,n=\pm 1$ at the detector stations in the beams a,a',b,b', under local phase settings $\phi_a,\phi_{a'},\phi_b,\phi_{b'}$, respectively, are given by

$$\frac{1}{4\sqrt{3}}[1+klmn\exp(i\Sigma\phi) + \tfrac{1}{2}(k\,e^{i\phi_a}+l\,e^{\phi_{a'}})(m\,e^{i\phi_b}+n\,e^{i\phi_{b'}})], \tag{9}$$

where $\Sigma\phi$ stands for the sum of all local phases. Therefore the probability of getting a particular set of results (k,l,m,n) is given by

$$\begin{aligned} P(k,l,m,n|\phi_a,\phi_{a'},\phi_b,\phi_{b'}) &= \frac{1}{16}\bigg(\frac{2}{3}(1+klmn\cos\Sigma\phi) \\ &\quad + \frac{1}{3}[1+kl\cos(\phi_a-\phi_{a'})][1+mn\cos(\phi_b-\phi_{b'})] \\ &\quad + \frac{1}{3}\mathrm{Re}\{[1+klmn\exp(i\Sigma\phi)] \\ &\quad \times(k\,e^{i\phi_a}+l\,e^{i\phi_{a'}})(m\,e^{i\phi_b}+n\,e^{\phi_{b'}})\}\bigg). \end{aligned} \tag{10}$$

The last term is written in the form of a real part of a complex function to shorten the expression.

The correlation function is defined as the mean value of the product of the four local results

$$E(\phi_a,\phi_{a'},\phi_b,\phi_{b'}) = \sum_{k=\pm1}\sum_{l=\pm1}\sum_{m=\pm1}\sum_{n=\pm1} klmn \times P(k,l,m,n|\phi_a,\phi_{a'},\phi_b,\phi_{b'}). \tag{11}$$

Its explicit form for the considered process is given by

$$E(\phi_a,\phi_{a'},\phi_b,\phi_{b'}) = \tfrac{2}{3}\cos\Sigma\phi + \frac{1}{3}\cos(\phi_a-\phi_{a'})\cos(\phi_b-\phi_{b'}). \tag{12}$$

Only the first two terms of the probabilities (10) contribute to the correlation function, and the function is itself a weighted sum of the GHZ correlation function (the first term) and a product of two Einstein-Podolsky-Rosen–Bell correlation functions. The last term in Eq. (10) gives a zero contribution to the correlation function, because sums like $\Sigma_{klmn} ln$ vanish.

The correlation function (12) for the process has a more complicated form than in the usual cases for GHZ-type states, but the strong correlations for numerous phase settings clearly indicate incompatibility with local realistic theories. When inserted into Mermin-type Bell inequalities [12] for four-particle systems, the violation is not too impressive, even for optimal sets of local phases. However, here we present a reasoning, involving Bell inequalities of an alternative type [13], giving stronger inequalities for distinguishing the validity of the different theories in a four photon experiment.

In a local hidden variable (LHV) theory a correlation function has to be modeled by a construction of the following form (see, e.g., [14,15]):

$$E_{LHV}(\phi_a,\phi_{a'},\phi_b,\phi_{b'}) = \int d\lambda\,\rho(\lambda) I_a(\phi_a,\lambda) I_{a'}(\phi_{a'},\lambda) \times I_b(\phi_b,\lambda) I_{b'}(\phi_{b'},\lambda), \tag{13}$$

where λ represents an arbitrary set of values of local hidden variables, $\rho(\lambda)$ their probabilistic distribution, and $I_x(\phi_x,\lambda)=\pm 1(x=a,a',b,b')$ represents the predetermined values of the measurements. Their values depend on the set of hidden variables and on the value of the *local* phase settings.

We start with allowing each observer of beam x $(=a,a',b,b')$ to choose, just like in the standard cases of the Bell and GHZ theorems [14,15], between two values ϕ_x^1 and ϕ_x^2 of the local phase settings.

The formula for the LHV correlation function for the chosen settings is given by

$$E_{LHV}(\phi_a^p,\phi_{a'}^q,\phi_b^r,\phi_{b'}^s)=\int d\lambda\,\rho(\lambda)I_a(\phi_a^p,\lambda)I_{a'}(\phi_{a'}^q,\lambda)\times I_b(\phi_b^r,\lambda)I_{b'}(\phi_{b'}^s,\lambda), \tag{14}$$

with $p,q,r,s=1,2$. It is important to stress that one must consider arbitrary LHV correlation functions. The only constraint being their structure given by Eq. (14).

One can treat the full set of the LHV predictions as a four-index tensor $\hat{E}_{LHV}$, with the indices $p,q,r,s=1,2$, built out of the tensorial products of two-dimensional real vectors $\mathbf{v}_x^\lambda=(I_x(\phi_x^1,\lambda),I_x(\phi_x^2,\lambda))$, which represent the two possible results of a given observer for the given value of the hidden variable:

$$\hat{E}_{LHV}=\int d\lambda\,\rho(\lambda)\mathbf{v}_a^\lambda\otimes\mathbf{v}_{a'}^\lambda\otimes\mathbf{v}_b^\lambda\otimes\mathbf{v}_{b'}^\lambda. \tag{15}$$

The actual values of the components of the two-dimensional vectors $(I_x(\phi_x^1,\lambda),I_x(\phi_x^2,\lambda))$ can be equal to only either (1,1), or $(1,-1)$, or $(-1,-1)$, or finally $(-1,1)$. Let us denote these four possible vectors by $\mathbf{v}_\mathbf{x}^\mathbf{j}$ with $j=1,2,3,4$, respectively. Thus, the LHV correlation function (tensor) can be simplified to a discrete sum over hidden probabilities $p_{k,l,m,n}$ of the tensorial products of all possible measurement results:

$$\hat{E}_{LHV}=\sum_{k,l,m,n=1,\ldots,4}p_{k,l,m,n}\mathbf{v}_\mathbf{a}^\mathbf{k}\otimes\mathbf{v}_{\mathbf{a}'}^\mathbf{l}\otimes\mathbf{v}_\mathbf{b}^\mathbf{m}\otimes\mathbf{v}_{\mathbf{b}'}^\mathbf{n}. \tag{16}$$

A further simplification of the tensor is possible since $(-1,-1)=-(1,1)$ and $(-1,1)=-(1,-1)$, or in other words $\mathbf{v}_\mathbf{x}^{\mathbf{k+2}}=-\mathbf{v}_\mathbf{x}^\mathbf{k}$. The tensorial products $\mathbf{v}_\mathbf{a}^\mathbf{k}\otimes\mathbf{v}_{\mathbf{a}'}^\mathbf{l}\otimes\mathbf{v}_\mathbf{b}^\mathbf{m}\otimes\mathbf{v}_{\mathbf{b}'}^\mathbf{n}$ with $k,l,m,n=1,2$ form a complete orthogonal (product) basis in the (real) Hilbert space of tensors $R^2\otimes R^2\otimes R^2\otimes R^2$. One can thus rewrite the expansion (16) so that it becomes an expansion in terms of the aforementioned basis

$$\hat{E}_{LHV}=\sum_{k,l,m,n=1,2}c_{k,l,m,n}\mathbf{v}_\mathbf{a}^\mathbf{k}\otimes\mathbf{v}_{\mathbf{a}'}^\mathbf{l}\otimes\mathbf{v}_\mathbf{b}^\mathbf{m}\otimes\mathbf{v}_{\mathbf{b}'}^\mathbf{n}. \tag{17}$$

The relation between the coefficients in Eq. (17) and the probabilities of Eq. (16) is given by

$$c_{k,l,m,n}=p_{k,l,m,n}-p_{k+2,l,m,n}-p_{k,l+2,m,n}-\cdots+p_{k+2,l+2,m,n}+\cdots-p_{k+2,l+2,m+2,n}+\cdots+p_{k+2,l+2,m+2,m+2}. \tag{18}$$

The expansion coefficients are of course *unique*, and since $\Sigma_{k,l,m,n=1,\ldots,4}\;p_{k,l,m,n}=1$, they satisfy the following inequality:

$$\sum_{k,l,m,n=1,2}|c_{k,l,m,n}|\leq 1. \tag{19}$$

This inequality is a necessary condition for the local realistic description to hold, and thus gives the handle for evaluating the validity of this class of theories. It should be stressed that one can also show that the inequality (19) is also a sufficient condition. This is due to the fact that the coefficients in Eq. (17) define the tensor E_{LHV} unambiguously (for details see [13], where the extension of this approach to an arbitrary number of qubits can be found).

To compare the structure of the possible LHV correlation functions with our quantum one (12), let us, in order to simplify the analysis, choose specific values for the local phase settings. First, the observer of beam a will be allowed the choice between $\phi_a^1=0$ and $\phi_a^2=\pi/2$. The other observers $(y=a',b,b')$ can choose between $\phi_y^1=-\pi/4$ and $\phi_y^2=\pi/4$. Next, one can expand the quantum function (12) into a sum of products of sine and cosine functions of *single* phases

$$\begin{aligned}E(\phi_a,\phi_{a'},\phi_b,\phi_{b'})&=c_ac_{a'}c_bc_{b'}+s_as_{a'}s_bs_{b'}\\&\quad-\tfrac{2}{3}(s_as_{a'}c_bc_{b'}+c_ac_{a'}s_bs_{b'})\\&\quad-\tfrac{2}{3}(s_ac_{a'}s_bc_{b'}+c_as_{a'}c_bs_{b'}\\&\quad+c_as_{a'}s_bc_{b'}+s_ac_{a'}c_bs_{b'}),\end{aligned} \tag{20}$$

where $s_x=\sin\phi_x$ and $c_x=\cos\phi_x$. For each fixed set of four local settings one can calculate the specific value of the quantum correlation function $E(\phi_a^p,\phi_{a'}^q,\phi_b^r,\phi_{b'}^s)$. We notice that for the specific phase settings given above one has $(y=a',b,b')$

$$(\cos(\phi_y^1),\cos(\phi_y^2))=\frac{1}{\sqrt{2}}(1,1)$$

and

$$(\sin(\phi_y^1),\sin(\phi_y^2))=-\frac{1}{\sqrt{2}}(1,-1),$$

whereas

$$(\cos(\phi_a^1),\cos(\phi_a^2))=(1,0)=\tfrac{1}{2}(1,1)+\tfrac{1}{2}(1,-1)$$

$$\text{and}\quad(\sin(\phi_a^1),\sin(\phi_a^2))=(0,1)=\tfrac{1}{2}(1,1)-\tfrac{1}{2}(1,-1).$$

Therefore the quantum predictions can be arranged to form a tensor, too, and it is easy to write down its expansion in the product basis [the same basis as in Eq. (17)]

$$\hat{E}=\sum_{k,l,m,n=1,2} q_{k,l,m,n}\mathbf{v}_{\mathbf{a}}^{\mathbf{k}}\otimes\mathbf{v}_{\mathbf{a}'}^{\mathbf{l}}\otimes\mathbf{v}_{\mathbf{b}}^{\mathbf{m}}\otimes\mathbf{v}_{\mathbf{b}'}^{\mathbf{n}}. \qquad (21)$$

The actual values of the expansion coefficients $q_{k,l,m,n}$ can be straightforwardly obtained from Eq. (20). However, note that for the specific set of angles chosen above one has

$$\sum_{k,l,m,n=1,2} |q_{k,l,m,n}|=\frac{8}{3\sqrt{2}}>1. \qquad (22)$$

Keeping in mind that the expansion in terms of basis vectors is unique; the quantum correlation function out of which the tensor $\hat{E}$ is built thus violates the necessary condition for local realism (19).

The quantum correlation function satisfies Eq. (19) *only if* it is multiplied by a scaling factor v equal to or smaller than $3\sqrt{2}/8\approx 53\%$, in other words, if one replaces it by

$$E'(\phi_a,\phi_{a'},\phi_b,\phi_{b'})=vE(\phi_a,\phi_{a'},\phi_b,\phi_{b'}). \qquad (23)$$

In an interferometry experiment this scaling parameter is directly related to the visibility (contrast) of the interference pattern. Visibilities lower than 1 can be interpreted as arising due to some noise contribution to the state. If one considers mixed states for the system of the type,

$$\rho_v=(1-v)\rho_{noise}+v|\boldsymbol{\psi}\rangle\langle\boldsymbol{\psi}|, \qquad (24)$$

where $\rho_{noise}=\frac{1}{16}\hat{I}$ represents completely uncorrelated noise contribution, and $|\boldsymbol{\psi}\rangle$ stands for our pure state (6), then the aforementioned critical v gives the threshold beyond which no LHV model can resemble the quantum predictions.

In conclusion, parametric type-II down-conversion not only produces entangled photon pairs, but also highly entangled four-photon states. The observation of these states is experimentally much easier to achieve than for GHZ-type states. Here, the full set of probabilities of possible LHV predictions is compared with the quantum predictions resulting in a significant distinction of the theories. The emission of more than two pairs will result in entangled states of more and more photons, although the correlation will decrease relative to the respective GHZ state.

This state enables implementation of various quantum communication schemes like quantum telecloning [16] or multiparty teleportation [17]. The other interesting feature of the considered state is that for a number of specific settings one obtains perfect four-photon correlations. E.g., for all local phases equal to zero the correlation function is equal to 1, whereas for $\phi_b=\phi_{b'}$, $\phi_a-\phi_{a'}=\pi$ and $\phi_{a'}+\phi_b=0$ it is equal to -1. This directly enables one to transfer the standard protocols for entanglement based quantum cryptography [18] to the four-photon case making multiparty quantum key distribution and quantum communication complexity schemes feasible.

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